Recall the nonhomogeneity $f(t)$ from earlier and imagine it to be non-zero.

Differential Equations
Class Notes
Nonhomogeneous Equations: The Method of Undetermined Coefficients (Section 4.4)

We have learned how to solve $ay''+by'+cy=f(t)$ where $f(t)=0$. But what if this $f(t)$ is *not* zero? In this section, this nonhomogeneity will be a single term of a certain type. It may seem crazy but we will *guess* at solutions to these equations. With some old-fashioned intuition, we can come up with a particular solution out of the infinite solutions that are out there.

Rationale and Method for Nonhomogeneous Equations:

Think about the equation $y''+3y'+2y=3t$. We must find a function $y(t)$ such that $y''+3y'+2y$ is a linear function of t (in this case, $3t$). What kind of function would fit? Perhaps a linear one?

Try $y(t)=At$ (where A is a real number). We would calculate $y'=A$ and $y''=0$. Put that all into the original equation and see if that works. Do you see a contradiction?

$y''+3y'+2y=3t$

$0+3A+2At=3t$

$2A=3$

$A=3/2$

$3A=0$

$A=0$

Wait. That doesn't work!

So, it does *not* work to simply use $y(t)=At$. Let's get a little more complicated in our guess. We'll try $y(t)=At+B$ (where A and B are real numbers). Calculate y' and y'' and see what you get when you put those into the original equation.

We end up with a system of linear equations we can solve for A and B. Solve it to write our solution $y(t)=At+B$.

$y=At+B$

$y'=A$

$y''=0$

$y''+3y'+2y=3t$

$0+3A+2(At+B)=3t$

$3A+2At+2B=3t$

$2A=3$

$A=3/2$

$3A+2B=0$

$3(3/2)+2B=0$

$9/2+2B=0$

$2B=-9/2$

$B=-9/4$

$\boxed{\text{So, } y_p=\frac{3}{2}t-\frac{9}{4}}$

This is called the method of undetermined coefficients because we assume the solution to be of a certain type but with unknown (or, yet to be determined) coefficients.

Method for a Certain Type of Nonhomogeneous Linear Equation:

For the equation $ay''+by'+cy=Ct^m$ where $m=0, 1, 2, \ldots$, we guess a particular solution in the form $y_p(t)=A_m t^m+\ldots+A_1 t+A_0$. These coefficients A_i are the **undetermined coefficients** we will find with a system of linear equations. In fact, we will solve a system of $m+1$ linear equations in $m+1$ unknowns.

When 0 is a root of the auxiliary equation, we have problems we will see later.

Note: We must retain all of the powers of $t^m, t^{m-1}, \ldots t^0$ in the proposed solution even though they may *not* appear in the nonhomogeneity $f(t)$.

expl 1: Find a particular solution to the diff. eq..

$2x'+x=3t^2$

Considering $f(t)=3t^2$, what is our proposed solution $x_p(t)$?

$m=2$

$$x_p(t)=A_2t^2+A_1t+A_0$$

$$x'=2A_2t+A_1$$

$$2x'+x=3t^2$$

$$4A_2t+2A_1+A_2t^2+A_1t+A_0=3t^2+0t+0$$

$A_2=3$

$$4A_2+A_1=0 \qquad 2A_1+A_0=0$$

$$12+A_1=0 \qquad -24+A_0=0$$

$$A_1=-12 \qquad A_0=24$$

$$\boxed{\text{Soln: } x_p(t)=3t^2-12t+24}$$

Different Types of Equations:

That is all well and good but that only covers one type of equation we will encounter. What about $y''+3y'+2y=f(t)$ where $f(t)=10e^{3t}$ or $f(t)=2\sin 5t$?

In the case of the above simple exponential nonhomogeneity, we will find that the function $y(t)=Ae^{3t}$ will suffice. (Try it out!) When $f(t)$ gets more complicated, we will need to get more creative.

In the case of $f(t)=2\sin 5t$, we need to remember that the derivative of sine is cosine. That implies the solution would include both sines and cosines, perhaps in the form $y(t)=A\sin 5t+B\cos 5t$.

Roots of Auxiliary Equations and Solutions to Nonhomogeneous Equations:

The examples so far have worked out. (Phew!) But there are equations that will yield nonsensical solutions and the problems are rooted (ah, a pun!) in the related auxiliary equations. Let's explore.

expl 2: Consider the diff. eq. $\boxed{y''+3y'+2y=10e^{-2t}}$ and its associated auxiliary equation $r^2+3r+2=0$. First, find the roots of the auxiliary equation. Then, show that the proposed solution $y(t)=Ae^{-2t}$ *cannot* be used to find a solution to the diff. eq. $y''+3y'+2y=10e^{-2t}$.

$$r^2+3r+2=0$$
$$(r+1)(r+2)=0$$
$$r=-1,\ -2$$

$$y_p(t)=Ae^{-2t}$$
$$y_p'=-2Ae^{-2t}$$
$$y_p''=4Ae^{-2t}$$

$$y''+3y'+2y=10e^{-2t}$$
$$4Ae^{-2t}-6Ae^{-2t}+2Ae^{-2t}=10e^{-2t}$$
$$0\cancel{=}10e^{-2t}$$

The problem that we are encountering here is dealt with a somewhat cumbersome theorem.

Method of Undetermined Coefficients for Certain Single-Term Nonhomogeneities:

To find a particular solution to the diff. eq. $ay'' + by' + cy = Ct^m e^{rt}$ where m is a non-negative integer, use the form $y_p(t) = t^s\left(A_m t^m + \ldots + A_1 t + A_0\right)e^{rt}$. We use the following values for s.

i.) Use $s = 0$ if r is *not* a root of the associated auxiliary equation.
ii.) Use $s = 1$ if r is a *simple* root of the associated auxiliary equation.
iii.) Use $s = 2$ if r is a *double* root of the associated auxiliary equation.

To find a particular solution to the diff. eq. $ay'' + by' + cy = \begin{cases} Ct^m e^{\alpha t}\cos\beta t \\ OR \\ Ct^m e^{\alpha t}\sin\beta t \end{cases}$ where β is non-zero, use the form

$$y_p(t) = t^s\left(A_m t^m + \ldots + A_1 t + A_0\right)e^{\alpha t}\cos\beta t + t^s\left(B_m t^m + \ldots + B_1 t + B_0\right)e^{\alpha t}\sin\beta t.$$

We use the following values for s.

iv.) Use $s = 0$ if $\alpha + i\beta$ is *not* a root of the associated auxiliary equation.
v.) Use $s = 1$ if $\alpha + i\beta$ is a *simple* root of the associated auxiliary equation.

expl 3: Decide if the method shown here can be used to solve the following equations. Explain.

a.) $y'' + 2y' - y = t^{-1}e^t$

Check the premises of the method above.

No, $m = -1$ and that's not allowed.

b.) $2y''(x) - 6y'(x) + y(x) = \dfrac{\sin x}{e^{4x}} = e^{-4x}\sin x$

yes, $\alpha = -4$ $\beta = 1$

c.) $y'' + 2y' - y = 4x\sin^2 x + 4x\cos^2 x$

$= 4x(\sin^2 x + \cos^2 x)$

$= 4x(1)$ yes, it fits "$Ct^m e^{0t}$" form

$= 4x$

$\frac{d}{dt}(\cos t) = -\sin(t)$

$\frac{d}{dt}(\sin t) = \cos(t)$

$3\sin 3t = C t^m e^{\alpha t} \sin \beta t$

$= 3 t^0 e^{0t} \sin(3t)$

expl 4: Find a particular solution to the diff. eq. below.

$y'' - y' + 9y = 3\sin(3t)$

What are m, α, and β? Is $\alpha + i\beta$ a root of the auxiliary equation?

$m = 0$

$\alpha = 0$

$\beta = 3$

$r^2 - r + 9 = 0$

$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{1 \pm \sqrt{1 - 4(1)(9)}}{2}$

$= \frac{1 \pm \sqrt{-35}}{2}$

$r = \frac{1}{2} \pm \frac{\sqrt{35}\, i}{2}$

roots of aux eqn.

$y_p(t) = \cancel{t^0}^{1} \cdot A \cdot \cancel{e^{0t}}^{1} \cos(3t) + \cancel{t^0}^{1} \cdot B \cdot \cancel{e^{0t}}^{1} \sin(3t)$

$y_p = A\cos(3t) + B\sin(3t)$

$y' = -3A\sin(3t) + 3B\cos(3t)$

$y'' = -9A\cos(3t) - 9B\sin(3t)$

$y'' - y' + 9y = 3\sin(3t)$

$-9A\cos(3t) - 9B\sin(3t)$ ← y''

$-3B\cos(3t) + 3A\sin(3t)$ ← $-y'$

$+9A\cos(3t) + 9B\sin(3t) = 3\sin(3t) + 0\cos(3t)$ ← $9y$

$-3B = 0$

$B = 0$

$3A = 3$

$A = 1$

$y_p(t) = 1\cos(3t)$

$$te^{2t} = Ct^m e^{rt}$$

$r=2$

$m=1$

expl 5: Find a particular solution to the diff. eq. below.

$x'' - 4x' + 4x = te^{2t}$

What are m and r? Show r to be a double root of the auxiliary equation.

$r^2 - 4r + 4 = 0$

$(r-2)(r-2) = 0$

$\boxed{r = 2 \text{ double root}} \to s=2$

$$x_p = t^2(A_1 t + A_0)e^{2t}$$

$$x_p = (A_1 t^3 + A_0 t^2)e^{2t}$$

$$x_p' = (A_1 t^3 + A_0 t^2)(2e^{2t}) + (3A_1 t^2 + 2A_0 t)e^{2t}$$

$$x'' = (A_1 t^3 + A_0 t^2)(4e^{2t}) + (3A_1 t^2 + 2A_0 t)(2e^{2t}) + (3A_1 t^2 + 2A_0 t)(2e^{2t}) + (6A_1 t + 2A_0)e^{2t}$$

$x'' - 4x' + 4x$

$= t^3 e^{2t}[\cancel{4A_1} - \cancel{8A_1} + \cancel{4A_1}] \to 0$

$+ t^2 e^{2t}[\cancel{4A_0} + \cancel{6A_1} + \cancel{6A_1} - \cancel{8A_0} - \cancel{12A_1} + \cancel{4A_0}] \to 0$

$+ te^{2t}[\cancel{4A_0} + \cancel{4A_0} + 6A_1 - \cancel{8A_0}]$

$+ e^{2t}[2A_0]$ $\qquad = te^{2t}$

$$6A_1 te^{2t} + 2A_0 e^{2t} = te^{2t} + 0e^{2t}$$

$6A_1 = 1$ $\qquad 2A_0 = 0$

$A_1 = 1/6$ $\qquad A_0 = 0$

$\boxed{x_p = \frac{1}{6}t^3 e^{2t}}$ is soln :)

Solving Higher-Order Linear Nonhomogeneous Equations:

We can extend our method for equations such as $2y''' + 3y'' + y' - 4y = e^{-t}$ or $y^{(4)} - 3y'' - 8y = \sin t$. We simply have to determine the auxiliary equation (using r^3 for y''', etc.) and determine if r (from the term e^{rt}) or $\alpha + i\beta$ (from $e^{\alpha t}\sin(\beta t)$) is a root.

expl 6: Find a particular solution to the higher-order diff. eq. below.

$2y''' + 3y'' + y' - 4y = e^{-t}$

$e^{rt}, r = -1$

You do *not* need to necessarily solve the auxiliary equation. Just determine if r is a root.

aux eqn:

$2r^3 + 3r^2 + r - 4 = 0$

Is $r = -1$ a root of aux eqn?

$2(-1)^3 + 3(-1)^2 + -1 - 4 \stackrel{?}{=} 0$

$-2 + 3 - 1 - 4 \stackrel{?}{=} 0$

No $\rightarrow$ so $r = -1$ is not a root of the aux eqn. $\rightarrow$ $s = 0$

$y_p = t^0(A_0)e^{-t}$

$y_p = Ae^{-t}$

$y' = -Ae^{-t}$

$y'' = Ae^{-t}$

$y''' = -Ae^{-t}$

$2y''' + 3y'' + y' - 4y$

$= -2Ae^{-t} + 3Ae^{-t} - Ae^{-t} - 4Ae^{-t} = e^{-t}$

$-4Ae^{-t} = e^{-t}$

$-4A = 1$

$A = -1/4$

$$\boxed{\therefore\ y_p = -\frac{1}{4}e^{-t}}$$