Recall the nonhomogeneity f(t) from earlier and imagine it to be non-zero.

Differential Equations

Class Notes

Nonhomogeneous Equations: The Method of Undetermined Coefficients (Section 4.4)

We have learned how to solve ay'' + by' + cy = f(t) where f(t) = 0. But what if this f(t) is not zero? In this section, this nonhomogeneity will be a single term of a certain type. It may seem crazy but we will guess at solutions to these equations. With some old-fashioned intuition, we can come up with a particular solution out of the infinite solutions that are out there.

### Rationale and Method for Nonhomogeneous Equations:

Think about the equation y'' + 3y' + 2y = 3t. We must find a function y(t) such that y'' + 3y' + 2y is a linear function of t (in this case, 3t). What kind of function would fit? Perhaps a linear one?

Try y(t) = At (where A is a real number). We would calculate y' = A and y'' = 0. Put that all into the original equation and see if that works. Do you see a contradiction?

$$y'' + 3y' + 2y = 3t$$
  
0 + 3A + 2At = 3t

A = 3 A = 0 A = 3/2 A = 0 A = 0

So, it does *not* work to simply use y(t) = At. Let's get a little more complicated in our guess. We'll try y(t) = At + B (where A and B are real numbers). Calculate y' and y'' and see what you get when you put those into the original equation.

We end up with a system of linear equations we can solve for A and B. Solve it to write our solution y(t) = At + B.

$$0 + 3A + 2(At+B) = 3t$$

A=3/2) 3 A+2B. 3 (3/2) + 2E

So, 4p = 3t - 9

9/2 + 2B = 0 2B = -9/2

This is called the method of undetermined coefficients because we assume the solution to be of a certain type but with unknown (or, yet to be determined) coefficients.

Method for a Certain Type of Nonhomogeneous Linear Equation:

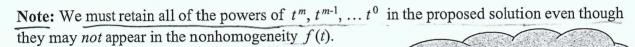
For the equation  $ay'' + by' + cy = Ct^m$  where m = 0, 1, 2, ..., we guess a particular solution in the form  $y_p(t) = A_m t^m + ... + A_1 t + A_0$ . These coefficients  $A_i$  are the **undetermined coefficients** we will find with a system of linear equations. In fact, we will solve a system of m + 1 linear equations in m + 1 unknowns.

When 0 is a root of the auxiliary equation, we have problems we will see later.

Considering  $f(t) = 3t^2$ .

what is our proposed

solution  $x_p(t)$ ?



expl 1: Find a particular solution to the diff. eq..

 $2x' + x = 3t^2$  $X_p(t) = A_2 t^2 + A_1 t + A_0$ 

 $X' = 2A_2t + A_1$ 

$$2x' + x = 3t^2$$

$$\begin{array}{c}
A_2 = 3 \\
A_2 = 3
\end{array}$$

$$\begin{array}{c}
4A_2 + A_1 = 0 \\
12 + A_1 = 0
\end{array}$$

$$\begin{array}{c}
7 - 24 + A_0 = 0 \\
A_0 = 24
\end{array}$$

#### **Different Types of Equations:**

That is all well and good but that only covers one type of equation we will encounter. What about y'' + 3y' + 2y = f(t) where  $f(t) = 10e^{3t}$  or  $f(t) = 2 \sin 5t$ ?

In the case of the above simple exponential nonhomogeneity, we will find that the function  $y(t) = Ae^{3t}$  will suffice. (Try it out!) When f(t) gets more complicated, we will need to get more creative.

In the case of  $f(t) = 2 \sin 5t$ , we need to remember that the derivative of sine is cosine. That implies the solution would include both sines and cosines, perhaps in the form  $y(t) = A \sin 5t + B \cos 5t$ .

## Roots of Auxiliary Equations and Solutions to Nonhomogeneous Equations:

The examples so far have worked out. (Phew!) But there are equations that will yield nonsensical solutions and the problems are rooted (ah, a pun!) in the related auxiliary equations. Let's explore.

expl 2: Consider the diff. eq.  $y'' + 3y' + 2y = 10e^{-2t}$  and its associated auxiliary equation  $r^2 + 3r + 2 = 0$ . First, find the roots of the auxiliary equation. Then, show that the proposed solution  $y(t) = Ae^{-2t}$  cannot be used to find a solution to the diff. eq.  $y'' + 3y' + 2y = 10e^{-2t}$ .

$$r^2 + 3r + 2 = 0$$
  $y_p(t) = A e^{-t}$   
 $(r+t)(r+2) = 0$   $y_p = -2Ae$   
 $r=-1,-2$   $y_p = 4Ae$ 

y'' + 3y' + 2y = 10e $4Ae^{-2t} - 6Ae^{-2t} + 2Ae^{-2t} = 10e^{-2t}$ 

The problem that we are encountering here is dealt with a somewhat cumbersome theorem.



# Method of Undetermined Coefficients for Certain Single-Term Nonhomogeneities:



To find a particular solution to the diff. eq.  $ay'' + by' + cy = Ct^m e^{rt}$  where m is a nonnegative integer, use the form  $y_p(t) = t^s (A_m t^m + ... + A_1 t + A_0) e^{rt}$ . We use the following values for s.



i.) Use s = 0 if r is not a root of the associated auxiliary equation.

- ii.) Use s = 1 if r is a *simple* root of the associated auxiliary equation.
- iii.) Use s = 2 if r is a *double* root of the associated auxiliary equation.



To find a particular solution to the diff. eq.  $ay'' + by' + cy = \begin{cases} Ct^m e^{\alpha t} \cos \beta t \\ OR \end{cases}$  where  $\beta$  is  $Ct^m e^{\alpha t} \sin \beta t$ 

non-zero, use the form

$$y_p(t) = t^s (A_m t^m + ... + A_1 t + A_0) e^{\alpha t} \cos \beta t + t^s (B_m t^m + ... + B_1 t + B_0) e^{\alpha t} \sin \beta t$$

We use the following values for s.



iv.) Use s = 0 if  $\alpha + i\beta$  is not a root of the associated auxiliary equation.

v.) Use s = 1 if  $\alpha + i\beta$  is a *simple* root of the associated auxiliary equation.



(expl 3: Decide if the method shown here can be used to solve the following equations. Explain.

a.) 
$$y'' + 2y' - y = t^{-1}e^{t}$$



Check the premises of the method above.

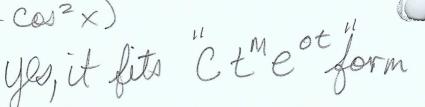
No, M = -1 and that's not allowed.

(b.) 
$$2y''(x) - 6y'(x) + y(x) = \frac{\sin x}{e^{4x}} = e^{-4x} \le \omega \times$$

$$(c.)y'' + 2y' - y = 4x \sin^2 x + 4x \cos^2 x$$

$$= 4\chi \left( Sin^2 \times + Cos^2 \times \right)$$

$$= 4 \times (1)$$



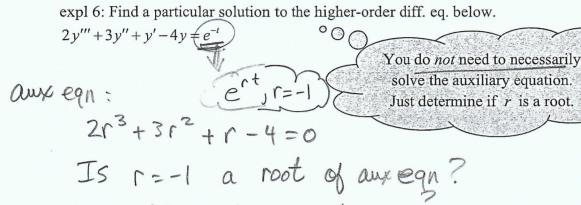
d (cost) = -sin(t) 3 sin3t = Ctm ext sin Bt = 3 t e ot sin (36) St(sint) = cos(t) M=0 expl 4: Find a particular solution to the diff. eq. below.  $y'' - y' + 9y = 3\sin(3t)$ What are m,  $\alpha$ , and  $\beta$ ? Is  $\alpha + i\beta$ a root of the auxiliary equation? yp(t) = to A: est cos (3t) + to B. est sin(3t) 4p = A cos (3t) + B sin (3t) = -3Asin (3t) + 3B cos (3t) -9A cos (3t) - 9B sin (3t) y'' - y' + 9y = 3 sin (3t)r= = = + \(\frac{35L}{35L}\) -9A cos(3t) - 9Bsin (3t) -3Bcos(3t) + 3Asin(3t) < +9Acos(3t) +9Bsin(3t) = 3sin(3t) (+0cos(3t) 3A = 3 -3B = 0yp(t)=1 cos(3t)

tezt = Ctmert expl 5: Find a particular solution to the diff. eq. below.  $x'' - 4x' + 4x = te^{2t}$ What are m and r? Show r to be a double root of the auxiliary equation  $p^2 - 4r + 4 = 0$ (r-2)(r-2)=0r=2 double root (-> (S=2)  $X_{p} = t^{2}(A_{1}t + A_{0})e^{2t}$ Xp = (A1t3+ Aota) e2t  $X_p = (A_1 t^3 + A_0 t^2)(2e^{2t}) + (3A_1 t^2 + 2A_0 t)e^{2t}$ X" = (A1t3 + Aot2) (4e2t) + (3A1t2 + 2Aot) (2e2t) + (3A1t2+2A0t)(2e2t)+ (6A1t+2A0) e2t X"-4x1+4x = t3e2t[4AI-8AI+4AI] + t2e2t [440 + 6A+ 6A+ -840=12A+440 +te2t[4A0+4A0+6A1-8A0] = te 2t + e2t [ 2Ao] 6AIte2+ + 2Aoe2+ = te2+ De2+ 6A1=1  $A_1 = 1/6$   $A_1$ 

## Solving Higher-Order Linear Nonhomogeneous Equations:

We can extend our method for equations such as  $2y''' + 3y'' + y' - 4y = e^{-t}$  or  $y^{(4)} - 3y'' - 8y = \sin t$ . We simply have to determine the auxiliary equation (using  $r^3$  for y''', etc.) and determine if r (from the term  $e^{rt}$ ) or  $\alpha + i\beta$  (from  $e^{\alpha t}\sin(\beta t)$ ) is a root.

expl 6: Find a particular solution to the higher-order diff. eq. below.



Is r=-1 a root of aux eqn?  $2(-1)^3 + 3(-1)^2 + -1 - 4 = 0$ <2 + 3 - 1 - 4 = 0 No -> so r=-1 is not a root of the aux egn. -> (5=0)

solve the auxiliary equation.

$$yp = t^{\circ}(A_{\circ})e^{-t}$$
 $yp = Ae^{-t}$ 
 $y'' = -Ae^{-t}$ 
 $y''' = Ae^{-t}$ 
 $y''' = Ae^{-t}$ 
 $y'''' = Ae^{-t}$ 
 $y'''' = -Ae^{-t}$ 
 $y'''' = -Ae^{-t}$ 
 $y'''' = -Ae^{-t}$ 
 $y'''' = -Ae^{-t}$ 
 $y'''' = -Ae^{-t}$ 

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